

# New Super Calogero Models and $\text{OSp}(4|2)$ Superconformal Mechanics <sup>1</sup>

Sergey Fedoruk, Evgeny Ivanov,

*Bogoliubov Laboratory of Theoretical Physics, JINR,  
141980, Dubna, Moscow Region, Russia  
fedoruk,eivanov@theor.jinr.ru*

Olaf Lechtenfeld

*Institut für Theoretische Physik, Leibniz Universität Hannover,  
Appelstraße 2, D-30167 Hannover, Germany  
lechtenf@itp.uni-hannover.de*

## Abstract

We report on the new approach to constructing superconformal extensions of the Calogero-type systems with an arbitrary number of involved particles. It is based upon the superfield gauging of non-abelian isometries of some supersymmetric matrix models. Among its applications, we focus on the new  $\mathcal{N}=4$  superconformal system yielding the  $\text{U}(2)$  spin Calogero model in the bosonic sector, and the one-particle case of this system, which is a new  $\text{OSp}(4|2)$  superconformal mechanics with non-dynamical  $\text{U}(2)$  spin variables. The characteristic feature of these models is that the strength of the conformal inverse-square potential is quantized.

## 1 Motivations and contents

The conformal Calogero model [1] describes  $n$  identical particles interacting pairwise through an inverse-square potential

$$V_C = \sum_{a \neq b} \frac{g}{(x_a - x_b)^2}, \quad a, b = 1, \dots, n. \quad (1)$$

It is a nice example of integrable  $d = 1$  system. This simplest  $(A_{n-1})$  Calogero model has some integrable generalizations, both conformal and non-conformal [1, 2].

As for superconformal extensions of the Calogero models (s-C models in what follows), the basic facts about them can be shortly summarized as follows:

- $\mathcal{N} = 2$  superextension of the model (1) and its some generalizations for any  $n$  was given by Freedman and Mende in 1990 [3] (see also [4] for  $\mathcal{N} = 1$  extensions).

---

<sup>1</sup>Talk presented by E. Ivanov at the XIII International Conference “Symmetry Methods in Physics”, Dubna, July 6-9, 2009.

- First attempts toward  $\mathcal{N} = 4$  extensions were undertaken by Wyllard in 2000 [5]. Further progress was achieved in refs. [6] - [10].
- Until recently,  $\mathcal{N}=4$  s-C models generalizing (1) for a generic  $n$  were not constructed.

At the same time, s-C systems are of great interest from various points of view. In 1999, Gibbons and Townsend [11] suggested that  $\mathcal{N} = 4$  s-C models might provide a microscopic description of the extreme Reissner-Nordström black hole in the near-horizon limit and, even more, be one of the faces of the hypothetical M-theory. Also, this sort of models can bear a tight relation to AdS/CFT and brane stuff (M-theory, strings, etc), quantum Hall effect (see, e.g., [2], [12] - [14]), etc. One-particle prototype of s-C systems is the superconformal mechanics. The first  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  variants of the latter were constructed and studied by Akulov and Pashnev in 1983 [15], Fubini and Rabinovici in 1984 [16] and Ivanov et al in 1989 [17]. These models attract a lot of attention mainly because of their intimate relationships to the description of the black-hole type solutions of supergravity (see e.g. [18, 19]).

Recently, we suggested a universal approach to s-C models for an arbitrary number  $n$  of interacting particles, including the  $\mathcal{N}=4$  models [20]. It is based on the superfield gauging of non-abelian isometries of some supersymmetric matrix models along the line of ref. [21].

*This new approach is based upon the following two primary principles:*

- $U(n)$  gauge invariance for  $n$ -particle s-C models;
- $\mathcal{N}$ -extended superconformal symmetry.

*The models constructed in this way display the following salient features:*

- Their bosonic sector is:
  - the standard  $A_{n-1}$  Calogero model for  $\mathcal{N}=1$  and  $\mathcal{N}=2$  cases,
  - a new variant of the  $U(2)$ -spin Calogero model [2] in the  $\mathcal{N}=4$  case;
- In the  $\mathcal{N}=2$  case there arise new superconformal extensions (different from those of Freedman and Mende);
- In the  $\mathcal{N}=4$  case the gauge approach directly yields  $OSp(4|2)$  as the superconformal group, but the general  $\mathcal{N}=4, d=1$  superconformal group  $D(2, 1; \alpha)$  can be incorporated as well;
- The center-of-mass coordinate in the  $\mathcal{N}=4$  case is not decoupled, and it acquires a conformal potential on shell. So a new model of  $\mathcal{N}=4$  superconformal mechanics emerges in the  $n=1$  limit [22, 23] (see also [24]).

In the present talk we give a brief account of this gauge approach, with the main focus on the  $\mathcal{N} = 4$  super Calogero model and the new  $\mathcal{N}=4$  superconformal mechanics just mentioned.

## 2 Bosonic Calogero as a gauge matrix model

The nice interpretation of the model (1) as a gauge model was given in [12, 13].

The starting point of this approach is the  $U(n)$ ,  $d=1$  matrix gauge theory which involves:

- an hermitian  $n \times n$ -matrix field  $X_a^b(t)$ ,  $(\overline{X_a^b} = X_b^a)$ ,  $(a, b = 1, \dots, n)$ ;
- a complex  $U(n)$ -spinor field  $Z_a(t)$ ,  $\bar{Z}^a = \overline{(Z_a)}$ ;
- $n^2$  non-propagating  $U(n)$  “gauge fields”  $A_a^b(t)$ ,  $(\overline{A_a^b} = A_b^a)$ .

The invariant action is written as [12]:

$$S_0 = \int dt \left[ \text{Tr} (\nabla X \nabla X) + \frac{i}{2} (\bar{Z} \nabla Z - \nabla \bar{Z} Z) + c \text{Tr} A \right], \quad (2)$$

where

$$\nabla X = \dot{X} + i[A, X], \quad \nabla Z = \dot{Z} + iAZ.$$

It respects the following invariances:

- The  $d = 1$  conformal  $SO(1, 2)$  invariance realized by the transformations:

$$\begin{aligned} \delta t &= a, & \partial_t^3 a &= 0, \\ \delta X_a^b &= \frac{1}{2} \dot{a} X_a^b, & \delta Z_a &= 0, & \delta A_a^b &= -\dot{a} A_a^b. \end{aligned}$$

- The invariance under the local  $U(n)$  transformations:

$$X \rightarrow gXg^\dagger, \quad Z \rightarrow gZ, \quad A \rightarrow gAg^\dagger + i\dot{g}g^\dagger,$$

where  $g(\tau) \in U(n)$ .

Using this gauge  $U(n)$  freedom, one can impose the following gauge conditions:

$$X_a^b = x_a \delta_a^b, \quad \bar{Z}^a = Z_a. \quad (3)$$

As the next step, one makes use of the algebraic equations of motion

$$\delta A_a^a : (Z_a)^2 = c, \quad \delta A_b^a \text{ (for } a \neq b) : A_a^b = \frac{Z_a Z_b}{2(x_a - x_b)^2}.$$

Substituting the expression for  $A_b^a$  back into the gauge-fixed form of the action (2), one recovers the standard Calogero action

$$S_0 = \int dt \left[ \sum_a \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{4(x_a - x_b)^2} \right].$$

Our approach to supersymmetric extensions of the Calogero models is just supersymmetrization of the above gauge approach, with the fields  $X, Z, A$  substituted by the appropriate  $d = 1$  superfields.

### 3 $\mathcal{N} = 1$ superconformal Calogero system

We start with a brief account of the  $\mathcal{N}=1, d=1$  supersymmetric version.

The point of departure in this case is the one-dimensional  $\mathcal{N} = 1$  supersymmetric  $U(n)$  gauge theory which involves:

- an even matrix superfield  $\mathcal{X}_a^b(t, \theta)$ ,  $(\mathcal{X})^\dagger = \mathcal{X}$ ,
- an even  $U(n)$ -spinor superfield  $\mathcal{Z}_a(t, \theta)$ ,  $\bar{\mathcal{Z}}^a(t, \theta) = (\mathcal{Z}_a)^\dagger$ ,
- an odd gauge connections  $\mathcal{A}_a^b(t, \theta)$ ,  $(\mathcal{A})^\dagger = -\mathcal{A}$ .

The invariant action is written as an integral over the  $\mathcal{N}=1, d=1$  superspace:

$$S_1 = -i \int dt d\theta \left[ \text{Tr} (\nabla_t \mathcal{X} \mathcal{D} \mathcal{X} + c \mathcal{A}) + \frac{i}{2} (\bar{\mathcal{Z}} \mathcal{D} \mathcal{Z} - \mathcal{D} \bar{\mathcal{Z}} \mathcal{Z}) \right], \quad (4)$$

where

$$\begin{aligned} \mathcal{D} \mathcal{X} &= D \mathcal{X} + i[\mathcal{A}, \mathcal{X}], \quad \nabla_t \mathcal{X} = -i \mathcal{D} \mathcal{D} \mathcal{X}, \quad \mathcal{D} \mathcal{Z} = D \mathcal{Z} + i \mathcal{A} \mathcal{Z}, \\ D &= \partial_\theta + i \theta \partial_t, \quad \{D, D\} = 2i \partial_t. \end{aligned}$$

The action (4) possesses  $\mathcal{N}=1$  superconformal  $OSp(1|2)$  invariance:

$$\begin{aligned} \delta t &= -i \eta \theta t, \quad \delta \theta = \eta t, \\ \delta \mathcal{X} &= -i \eta \theta \mathcal{X}, \quad \delta \mathcal{A} = i \eta \theta \mathcal{A}, \quad \delta \mathcal{Z} = 0, \end{aligned}$$

and gauge  $U(n)$  invariance:

$$\mathcal{X}' = e^{i\tau} \mathcal{X} e^{-i\tau}, \quad \mathcal{Z}' = e^{i\tau} \mathcal{Z}, \quad \mathcal{A}' = e^{i\tau} \mathcal{A} e^{-i\tau} - i e^{i\tau} D e^{-i\tau},$$

where  $\tau_a^b(t, \theta) \in u(n)$  is an hermitian matrix parameter.

One can choose WZ gauge for the spinor connection:

$$\mathcal{A} = i \theta A(t). \quad (5)$$

After integrating over  $\theta$  s in the gauge-fixed form of (4) and eliminating auxiliary fields, one obtains:

$$S_1 = S_0 + S_1^\Psi, \quad S_1^\Psi = -i \text{Tr} \int dt \Psi \nabla \Psi, \quad (6)$$

where  $\Psi = -i D \mathcal{X}$  and  $\nabla \Psi = \dot{\Psi} + i[A, \Psi]$ . The bosonic part  $S_0$  of  $S_1$  in (6) is just the “gauge-unfixed” Calogero action (2).

After integrating out non-propagating gauge fields  $A_b^a$  from the total action  $S_1$  and fixing the residual  $U(n)$  gauge freedom of the WZ gauge (5) in the same way as in (3), we obtain an  $\mathcal{N}=1$  superconformal action which contains  $n$  bosonic fields  $x_a$  with the standard conformal Calogero potential (1) accompanied by interactions with  $n^2$  physical fermionic fields  $\psi_b^a$ .

## 4 $\mathcal{N} = 4$ superconformal Calogero

$\mathcal{N}=4, d=1$  models are naturally formulated in the  $d = 1$  version of harmonic superspace (HSS) [25] :

$$(t, \theta_i, \bar{\theta}^k, u_i^\pm), \quad i, k = 1, 2.$$

Bosonic SU(2)-doublets  $u_i^\pm$  are harmonic coordinates, with the basic relation  $u^{+i}u_i^- = 1$ . The main feature of HSS is the presence of harmonic analytic subspace in it (an analog of chiral superspace), closed under the action of  $\mathcal{N}=4$  supersymmetry:

$$\begin{aligned} (\zeta, u) &= (t_A, \theta^+, \bar{\theta}^+, u_i^\pm), \\ t_A &= t - i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+), \quad \theta^\pm = \theta^i u_i^\pm, \quad \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm. \end{aligned}$$

The integration measures in the full HSS and its analytic subspace are defined, respectively, as:

$$\mu_H = du dt d^4 \theta, \quad \mu_A^{(-2)} = du d\zeta^{(-2)}.$$

The  $\mathcal{N} = 4, d = 1$  supergauge theory which generalizes the bosonic and  $\mathcal{N}=1$  examples described above involves the following superfields:

- The hermitian matrix superfields  $\mathcal{X} = (\mathcal{X}_a^b)$  (multiplets **(1,4,3)**):

$$\mathcal{D}^{++} \mathcal{X} = 0, \quad \mathcal{D}^+ \mathcal{D}^- \mathcal{X} = 0, \quad (\mathcal{D}^+ \bar{\mathcal{D}}^- + \bar{\mathcal{D}}^+ \mathcal{D}^-) \mathcal{X} = 0;$$

- Analytic superfields  $\mathcal{Z}_a^+(\zeta, u)$  (multiplets **(4,4,0)**):

$$\mathcal{D}^{++} \mathcal{Z}^+ = 0, \quad D^+ \mathcal{Z}^+ = 0, \quad \bar{D}^+ \mathcal{Z}^+ = 0;$$

- The analytic gauge matrix connection  $V^{++}(\zeta, u)$ . It specifies the gauge-covariant derivatives (harmonic and spinor):

$$\mathcal{D}^{++} \mathcal{Z}^+ = (D^{++} + i V^{++}) \mathcal{Z}^+, \quad \mathcal{D}^{++} \mathcal{X} = D^{++} \mathcal{X} + i [V^{++}, \mathcal{X}], \text{ etc.}$$

The  $\mathcal{N}=4$  superconformal action is a sum of three terms:

$$S_4 = S_{\mathcal{X}} + S_{WZ} + S_{FI}, \tag{7}$$

where

$$S_{\mathcal{X}} = -\frac{1}{2} \int \mu_H \text{Tr}(\mathcal{X}^2), \quad S_{WZ} = \frac{1}{2} \int \mu_A^{(-2)} \mathcal{V}_0 \tilde{\mathcal{Z}}^+ \mathcal{Z}^+, \tag{8}$$

$$S_{FI} = \frac{i}{2} c \int \mu_A^{(-2)} \text{Tr} V^{++} \tag{9}$$

and  $\mathcal{V}_0(\zeta, u)$  is a real analytic superfield, which is related to  $\mathcal{X}_0 \equiv \text{Tr}(\mathcal{X})$  by the integral transform

$$\begin{aligned} \mathcal{X}_0(t, \theta_i, \bar{\theta}^i) &= \int du \mathcal{V}_0(t_A, \theta^+, \bar{\theta}^+, u^\pm) \Big|_{\theta^\pm = \theta^i u_i^\pm, \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm}, \\ \mathcal{V}_0' &= \mathcal{V}_0 + D^{++} \Lambda^{--}. \end{aligned} \tag{10}$$

The action (7) respects the following set of invariances:

- $\mathcal{N}=4$  superconformal invariance under the supergroup  $D(2, 1; \alpha = -1/2) \simeq \text{OSp}(4|2)$ :

$$\delta \mathcal{X} = -\Lambda_0 \mathcal{X}, \quad \delta \mathcal{Z}^+ = \Lambda \mathcal{Z}^+, \quad \delta V^{++} = 0, \quad \delta \mathcal{V}_0 = -2\Lambda \mathcal{V}_0,$$

$$\Lambda = 2i\alpha(\bar{\eta}^-\theta^+ - \eta^-\bar{\theta}^+), \quad \Lambda_0 = 2\Lambda - D^{--}D^{++}\Lambda;$$

- Gauge  $U(n)$  invariance:

$$\mathcal{X}' = e^{i\lambda} \mathcal{X} e^{-i\lambda}, \quad \mathcal{Z}^{+'} = e^{i\lambda} \mathcal{Z}^+,$$

$$V^{++'} = e^{i\lambda} V^{++} e^{-i\lambda} - i e^{i\lambda} (D^{++} e^{-i\lambda}),$$

where  $\lambda_a^b(\zeta, u^\pm) \in u(n)$  is the ‘hermitian’ analytic matrix parameter,  $\tilde{\lambda} = \lambda$ .

Like in the  $\mathcal{N}=1$  case, using the  $U(n)$  gauge freedom we can choose the WZ gauge for  $V^{++}$ :

$$V^{++} = -2i \theta^+ \bar{\theta}^+ A(t_A). \quad (11)$$

In this gauge:

$$S_4 = S_b + S_f, \quad (12)$$

$$S_b = \int dt \left[ \text{Tr}(\nabla X \nabla X + c A) + \frac{i}{2} X_0 (\bar{Z}_k \nabla Z^k - \nabla \bar{Z}_k Z^k) + \frac{n}{8} (\bar{Z}^{(i} Z^{k)}) (\bar{Z}_i Z_k) \right],$$

$$S_f = -i \text{Tr} \int dt (\bar{\Psi}_k \nabla \Psi^k - \nabla \bar{\Psi}_k \Psi^k) - \int dt \frac{\Psi_0^{(i} \bar{\Psi}_0^{k)} (\bar{Z}_i Z_k)}{X_0}.$$

Here  $\mathcal{X} = X(t_A) + \theta^- \Psi^i(t_A) u_i^+ + \bar{\theta}^- \bar{\Psi}^i(t_A) u_i^+ + \dots$ ,  $X_0 \equiv \text{Tr}(X)$ ,  $\Psi_0^i \equiv \text{Tr}(\Psi^i)$ ,  $\bar{\Psi}_0^i \equiv \text{Tr}(\bar{\Psi}^i)$ ,  $\mathcal{Z}^+ = Z^i(t_A) u_i^+ + \dots$ .

Let us study the bosonic limit of the action (12). To pass to this limit, one needs:

- to impose the gauge  $X_a^b = 0$ ,  $a \neq b$ ;
- to eliminate  $A_a^b$ ,  $a \neq b$ , by their algebraic equations of motion;
- to pass to the new fields  $Z'^i_a = (X_0)^{1/2} Z^i_a$  (in what follows, we shall omit primes).

As a result, we obtain the following bosonic action

$$S_b = \int dt \left\{ \sum_a \dot{x}_a \dot{x}_a + \frac{i}{2} \sum_a (\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) + \sum_{a \neq b} \frac{\text{Tr}(S_a S_b)}{4(x_a - x_b)^2} - \frac{n \text{Tr}(\hat{S} \hat{S})}{2(X_0)^2} \right\}, \quad (13)$$

where

$$(S_a)_i^j \equiv \bar{Z}_i^a Z_a^j, \quad (\hat{S})_i^j \equiv \sum_a [(S_a)_i^j - \frac{1}{2} \delta_i^j (S_a)_k^k]. \quad (14)$$

The fields  $Z_a^k$  are subject to the constraints

$$\bar{Z}_i^a Z_a^i = c \quad \forall a. \quad (15)$$

Since the fields  $Z_k^a, \bar{Z}_k^a$  are described by the Lagrangian of the first order in the time derivative, we are led quantize them by the Dirac quantization method:

$$\frac{i}{2} \int dt \sum_a (\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) \quad \Rightarrow \quad [\bar{Z}_i^a, Z_b^j]_D = i\delta_b^a \delta_i^j.$$

Now it is easy to check that the quantities  $S_a$  defined in (14) form, for each  $a$ ,  $u(2)$  algebras

$$[(S_a)_i^j, (S_b)_k^l]_D = i\delta_{ab} \{ \delta_i^l (S_a)_k^j - \delta_k^j (S_a)_i^l \}.$$

Modulo center-of-mass conformal potential, the bosonic action (13) can be written as

$$S'_b = \int dt \left\{ \sum_a \dot{x}_a \dot{x}_a + \sum_{a \neq b} \frac{\text{Tr}(S_a S_b)}{4(x_a - x_b)^2} \right\}. \quad (16)$$

It is none other than the action of integrable  $U(2)$ -spin Calogero model [2].

## 5 $OSp(4|2)$ superconformal mechanics

The  $n = 1$  case of the  $\mathcal{N}=4$  Calogero model, as distinct from the  $\mathcal{N}=1$  and  $\mathcal{N}=2$  models, already yields a conformal inverse-square potential at the component level (for the center-of-mass coordinate) and so yields a non-trivial  $\mathcal{N}=4$  superconformal mechanics model [22].

The corresponding superfield action is

$$S = S_{\mathcal{X}} + S_{FI} + S_{WZ}, \quad (17)$$

where

$$\begin{aligned} S_{\mathcal{X}} &= -\frac{1}{2} \int \mu_H \mathcal{X}^2, \quad S_{FI} = \frac{i}{2} c \int \mu_A^{(-2)} V^{++}, \quad S_{WZ} = \frac{1}{2} \int \mu_A^{(-2)} \mathcal{V} \bar{\mathcal{Z}}^+ \mathcal{Z}^+, \\ D^{++} \mathcal{X} &= 0, \quad D^+ D^- \mathcal{X} = \bar{D}^+ \bar{D}^- \mathcal{X} = (D^+ \bar{D}^- + \bar{D}^+ D^-) \mathcal{X} = 0, \\ \mathcal{D}^{++} \mathcal{Z}^+ &\equiv (D^{++} + i V^{++}) \mathcal{Z}^+ = 0, \quad \mathcal{D}^{++} \bar{\mathcal{Z}}^+ \equiv (D^{++} - i V^{++}) \bar{\mathcal{Z}}^+ = 0, \\ \delta V^{++} &= -D^{++} \lambda, \quad \delta \mathcal{Z}^+ = i \lambda \mathcal{Z}^+, \quad \delta \mathcal{V} = D^{++} \lambda^{--}. \end{aligned} \quad (18)$$

The actions  $S_{FI}$  and  $S_{WZ}$  are invariant under the general  $\mathcal{N}=4$  superconformal group  $D(2, 1; \alpha)$  for arbitrary  $\alpha$ , while  $S_{\mathcal{X}}$  - only with respect to  $D(2, 1; \alpha = -1/2) \sim OSp(4|2)$ , so the total action is  $OSp(4|2)$ -superconformal.

The action (17) can be generalized to any  $\alpha$ , but at cost of nonlinear action for  $\mathcal{X}$ . So we limit our presentation here to the case of  $\alpha = -1/2$ .

The off-shell component content of the model is  $(\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{4}, \mathbf{4}, \mathbf{0}) \oplus V^{++} = (\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{3}, \mathbf{4}, \mathbf{1})$ . So it is reducible. The on-shell content can be most clearly revealed in the WZ gauge  $V^{++} = -2i\theta^+ \bar{\theta}^+ A(t)$ . After eliminating auxiliary fields from the total action, the latter takes the form

$$S = S_b + S_f, \quad (19)$$

where

$$\begin{aligned} S_b &= \int dt \left[ \dot{x}\dot{x} + \frac{i}{2} (\bar{z}_k \dot{z}^k - \dot{\bar{z}}_k z^k) - \frac{(\bar{z}_k z^k)^2}{16x^2} - A (\bar{z}_k z^k - c) \right], \\ S_f &= -i \int dt \left( \bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \psi^k \right) - \int dt \frac{\psi^i \bar{\psi}^k z_{(i} \bar{z}_{k)}}{x^2}. \end{aligned} \quad (20)$$

Varying with respect to  $A(t)$  as a Lagrange multiplier yields the constraint

$$\bar{z}_k z^k = c. \quad (21)$$

After properly fixing the residual U(1) gauge invariance, one can solve this constraint in terms of two independent fields  $\gamma(t)$  and  $\alpha(t)$  as

$$z^1 = \kappa \cos \frac{\gamma}{2} e^{i\alpha/2}, \quad z^2 = \kappa \sin \frac{\gamma}{2} e^{-i\alpha/2}, \quad \kappa^2 = c. \quad (22)$$

Then the bosonic action takes the form

$$S_b = \int dt \left[ \dot{x}\dot{x} - \frac{c^2}{16x^2} - \frac{c}{2} \cos \gamma \dot{\alpha} \right]. \quad (23)$$

It is a sum of the standard conformal mechanics action for the variable  $x(t)$ , and the  $S^2$  Wess-Zumino term for the angular variables  $\gamma(t)$  and  $\alpha(t)$ , of the first-order in time derivative. Taken separately, this term provides an example of Chern-Simons mechanics [26, 27]. The variables  $\gamma(t)$  and  $\alpha(t)$  (or  $z^k$  and  $\bar{z}_k$  in the manifestly SU(2) covariant formulation) become spin degrees of freedom (“target harmonics”) upon quantization.

We quantize by Dirac procedure,

$$\begin{aligned} [X, P] &= i, \quad [Z^i, \bar{Z}_j] = \delta_j^i, \quad \{\Psi^i, \bar{\Psi}_j\} = -\frac{1}{2} \delta_j^i, \\ P &= \frac{1}{i} \partial / \partial X, \quad \bar{Z}_i = v_i^+, \quad Z^i = \partial / \partial v_i^+, \quad \Psi^i = \psi^i, \quad \bar{\Psi}_i = -\frac{1}{2} \partial / \partial \psi^i. \end{aligned} \quad (24)$$

The wave function is subject to the constraint

$$D^0 \Phi = \bar{Z}_i Z^i \Phi = v_i^+ \frac{\partial}{\partial v_i^+} \Phi = c \Phi, \quad (25)$$

whence

$$\Phi = A_1^{(c)} + \psi^i B_i^{(c)} + \psi^i \psi_i A_2^{(c)}, \quad (26)$$

and the component fields are collections of the SU(2) irreps with the isospins  $c/2$ ,  $(c+1)/2$ ,  $(c-1)/2$  and  $c/2$ , respectively (the component fields depend on  $X$ ).

The constant  $c$  gets quantized,  $c \in \mathbb{Z}$ , as a consequence of requiring the wave function (26) with the constraint (25) to be single-valued. The same phenomenon takes place in the general  $n$ -particle  $\mathcal{N}=4$  Calogero system due to the constraints (15) which, after quantization, become analogs of (25). This is also in agreement with the analogous

arguments in the topological Chern–Simons quantum mechanics [27], which are based upon the path integral quantization.<sup>2</sup>

The quantum Hamiltonian is given by the expression

$$\mathbf{H} = \frac{1}{4} \left( P^2 + \frac{l(l+1)}{X^2} \right), \quad l = (c/2, (c+1)/2, (c-1)/2, c/2). \quad (27)$$

The basic distinguishing feature of the new superconformal mechanics model is that the bosonic sector of the space of its quantum states (with fermionic states neglected) is a direct product of the space of states of the standard conformal mechanics (parametrized by  $X$ ) and a fuzzy sphere [29] (parametrized by  $Z^i, \bar{Z}_i$ ). The full wave functions are irreps of  $SU(2)$ . The whole space of states (with fermions), shows no any product structure.

One can ask what is the brane analog of this new superconformal mechanics via  $AdS_2/CFT_1$  correspondence. The preliminary answer is that it is some superparticle evolving on  $AdS_2$  and coupled to the external magnetic charge via WZ term.

## 6 Summary and outlook

Let us summarize the results presented in the Talk and outline directions of the further studies.

- We proposed a new gauge approach to the construction of superconformal Calogero-type systems. The characteristic features of this approach are the presence of auxiliary supermultiplets with WZ type actions, the built-in superconformal invariance and the emergence of the Calogero coupling constant as a strength of the FI term of the  $U(1)$  gauge (super)field. This strength is quantized.
- We used the  $U(n)$  gauging and obtained superextensions of the  $A_{n-1}$  Calogero model. Superextensions of other conformal Calogero models could be obtained by choosing other gauge groups.
- The  $\mathcal{N}=4$  action presented is invariant under  $D(2, 1; -1/2) \cong OSp(4|2)$ . It can be easily generalized to an arbitrary  $\alpha$ .<sup>3</sup>
- We constructed a new  $\mathcal{N}=4$  superconformal mechanics with the  $OSp(4|2)$  invariance as the extreme  $n = 1$  case of our  $\mathcal{N}=4$  Calogero system. After quantization it yields a fuzzy sphere in the bosonic sector.

As the mainstream directions of the future work we would like to distinguish **(i)** construction of the full quantum version of the new  $\mathcal{N} = 4$  super Calogero model for any

---

<sup>2</sup>Actually, in the considered gauge approach to Calogero-type models the constant  $c$  is quantized in all cases, including the purely bosonic one, due to its appearance as a strength of the term  $c \int dt \text{Tr} A$  in the total component actions [12] (see also a recent paper [28]).

<sup>3</sup>For the particular case of  $n = 1$  (superconformal mechanics) such a generalization was given in a recent paper [23]. For  $\alpha \neq 0$ , the general  $n$  particle action was given in [20].

number of particles (as well as of the new  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  extensions); **(ii)** elucidating the relationships of this  $\mathcal{N} = 4$  super Calogero system to the  $\mathcal{N} = 4$  Calogero-type systems considered in [5] - [10] and to the black-hole stuff; **(iii)** analysis of possible integrability properties of the new super Calogero systems (searching for their Lax pair representation, etc).

## Acknowledgements

We acknowledge a support from a DFG grant, project No 436 RUS/113/669 (E.I. & O.L.), the RFBR grants 08-02-90490, 09-02-01209 and 09-01-93107 (S.F. & E.I.) and a grant of the Heisenberg-Landau Program.

## References

- [1] F. Calogero, *Solution of a three-body problem in one-dimension*, J.Math.Phys. **10** (1969) 2191; *Ground state of one-dimensional  $N$  body system*, J.Math.Phys. **10** (1969) 2197.
- [2] A.P. Polychronakos, *The physics and mathematics of Calogero particles*, J.Phys. **A39** (2006) 12793.
- [3] D.Z. Freedman, P.F. Mende, *An Exactly Solvable  $N$  Particle System In Supersymmetric Quantum Mechanics*, Nucl.Phys. **B344** (1990) 317.
- [4] L. Brink, T.H. Hansson, M.A. Vasiliev, *Explicit solution to the  $N$  body Calogero problem*, Phys.Lett. **B286** (1992) 109, [arXiv:hep-th/9206049](#); L. Brink, T.H. Hansson, S. Konstein, M.A. Vasiliev, *Anyonic representation, fermionic extension and supersymmetry*, Nucl.Phys. **B401** (1993) 591, [arXiv:hep-th/9302023](#).
- [5] N. Wyllard, *(Super)conformal many body quantum mechanics with extended supersymmetry*, J.Math.Phys. **41** (2000) 2826, [arXiv:hep-th/9910160](#).
- [6] S. Bellucci, A. Galajinsky, S. Krivonos, *New many-body superconformal models as reductions of simple composite systems*, Phys.Rev. **D68** (2003) 064010, [arXiv:hep-th/0304087](#).
- [7] S. Bellucci, A.V. Galajinsky, E. Latini, *New insight into WDVV equation*, Phys.Rev. **D71** (2005) 044023, [arXiv:hep-th/0411232](#).
- [8] A. Galajinsky, O. Lechtenfeld, K. Polovnikov, *Calogero models and nonlocal conformal transformations*, Phys.Lett. **B643** (2006) 221-227, [arXiv:hep-th/0607215](#);  *$N=4$  superconformal Calogero models*, JHEP 0711 (2007) 008, [arXiv:0708.1075 \[hep-th\]](#);  *$N=4$  mechanics, WDVV equations and roots*, JHEP **0903** (2009) 113, [arXiv:0802.4386 \[hep-th\]](#).

- [9] S. Bellucci, S. Krivonos, A. Sutulin,  *$N=4$  supersymmetric 3-particles Calogero model*, Nucl.Phys. **B805** (2008) 24, [arXiv:0805.3480 \[hep-th\]](#).
- [10] S. Krivonos, O. Lechtenfeld, K. Polovnikov,  *$N=4$  superconformal  $n$ -particle mechanics via superspace*, Nucl.Phys. **B817** (2009) 265, [arXiv:0812.5062 \[hep-th\]](#).
- [11] G.W. Gibbons, P.K. Townsend, *Black holes and Calogero models*, Phys.Lett. **B454** (1999) 187, [arXiv:hep-th/9812034](#).
- [12] A.P. Polychronakos, *Integrable systems from gauged matrix models*, Phys.Lett. **B266** (1991) 29.
- [13] A. Gorsky, N. Nekrasov, *Quantum integrable systems of particles as gauge theories*, Theor.Math.Phys. **100** (1994) 874 [Teor.Mat.Fiz. **100** (1994) 97]; *Relativistic Calogero-Moser model as gauged WZW theory*, Nucl.Phys. **B436** (1995) 582, [arXiv:hep-th/9401017](#).
- [14] A.P. Polychronakos, *Quantum Hall states as matrix Chern-Simons theory*, JHEP **0104** (2001) 011, [arXiv:hep-th/0103013](#); B. Morariu, A.P. Polychronakos, *Finite noncommutative Chern-Simons with a Wilson line and the quantum Hall effect*, JHEP **0107** (2001) 006, [arXiv:hep-th/0106072](#); *Fractional quantum Hall effect on the two-sphere: A Matrix model proposal*, Phys.Rev. **D72** (2005) 125002, [arXiv:hep-th/0510034](#).
- [15] V.P. Akulov, A.I. Pashnev, *Quantum superconformal model in  $(1,2)$  space*, Theor.Math.Phys. **56** (1983) 344.
- [16] S. Fubini, E. Rabinovici, *Superconformal quantum mechanics*, Nucl.Phys. **B245** (1984) 17.
- [17] E.A. Ivanov, S.O. Krivonos, V.M. Leviant, *Geometric superfield approach to superconformal mechanics*, J.Phys. **A22** (1989) 4201.
- [18] P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend, A. Van Proeyen, *Black holes and superconformal mechanics*, Phys.Rev.Lett. **81** (1998) 4553, [arXiv:hep-th/9804177](#); J.A. de Azcarraga, J.M. Izquierdo, J.C. Perez Bueno, P.K. Townsend, *Superconformal mechanics and nonlinear realizations*, Phys.Rev. **D59** (1999) 084015, [arXiv:hep-th/9810230](#); E. Ivanov, S. Krivonos, J. Niederle, *Conformal and superconformal mechanics revisited*, Nucl.Phys. **B677** (2004) 485, [arXiv:hep-th/0210196](#); S. Bellucci, A. Galajinsky, E. Ivanov, S. Krivonos,  *$AdS(2)/CFT(1)$ , canonical transformations and superconformal mechanics*, Phys.Lett. **B555** (2003) 99, [arXiv:hep-th/0212204](#).
- [19] J. Michelson, A. Strominger, *The geometry of (super)conformal quantum mechanics*, Commun.Math.Phys. **213** (2000) 1, [arXiv:hep-th/9907191](#); *Superconformal multi-black hole quantum mechanics*, JHEP **9909** (1999) 005, [arXiv:hep-th/9908044](#); A. Maloney, M. Spradlin, A. Strominger, *Superconformal multi-black hole moduli*

- spaces in four dimensions*, JHEP **0204** (2002) 003, [arXiv:hep-th/9911001](#); G. Papadopoulos, *Conformal and superconformal mechanics*, Class.Quant.Grav. **17** (2000) 3715, [arXiv:hep-th/0002007](#).
- [20] S. Fedoruk, E. Ivanov, O. Lechtenfeld, *Supersymmetric Calogero models by gauging*, Phys.Rev. **D79** (2009) 105015, [arXiv:0812.4276 \[hep-th\]](#).
  - [21] F. Delduc, E. Ivanov, *Gauging  $N=4$  supersymmetric mechanics*, Nucl.Phys. **B753** (2006) 211, [arXiv:hep-th/0605211](#).
  - [22] S. Fedoruk, E. Ivanov, O. Lechtenfeld,  *$OSp(4|2)$  superconformal mechanics*, JHEP **0908** (2009) 081, [arXiv:0905.4951 \[hep-th\]](#).
  - [23] S. Fedoruk, E. Ivanov, O. Lechtenfeld, *New  $D(2,1;\alpha)$  Mechanics with Spin Variables*, [arXiv:0912.3508 \[hep-th\]](#).
  - [24] S. Krivonos, O. Lechtenfeld,  *$SU(2)$  reduction in  $N=4$  supersymmetric mechanics*, Phys.Rev. **D80** (2009) 045019, [arXiv:0906.2469 \[hep-th\]](#).
  - [25] E. Ivanov, O. Lechtenfeld,  *$N=4$  supersymmetric mechanics in harmonic superspace*, JHEP **0309** (2003) 073, [arXiv:hep-th/0307111](#).
  - [26] L. Faddeev, R. Jackiw, *Hamiltonian Reduction of Unconstrained and Constrained Systems*, Phys.Rev.Lett. **60** (1988) 1692; F. Roberto, R. Percacci, E. Sezgin, *Sigma Models With Purely Wess-Zumino-Witten Actions*, Nucl.Phys. **B322** (1989) 255; G.V. Dunne, R. Jackiw, C.A. Trugenberger, *“Topological” (Chern-Simons) quantum mechanics*, Phys.Rev. **D41** (1990) 661.
  - [27] P.S. Howe, P.K. Townsend, *Chern-Simons Quantum Mechanics*, Class.Quant.Grav. **7** (1990) 1655.
  - [28] E.A. Ivanov, M.A. Konyushikhin, A.V. Smilga, *SQM with Non-Abelian Self-Dual Fields: Harmonic Superspace Description*, [arXiv:0912.3289 \[hep-th\]](#).
  - [29] J. Madore, *Quantum mechanics on a fuzzy sphere*, Phys.Lett. **B263** (1991) 245; *The fuzzy sphere*, Class.Quant.Grav. **9** (1992) 69.